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Tilings in Lee metric

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ABSTRACT

Gravier et al. proved [S. Gravier, M. Mollard, Ch. Payan, On the existence of three-dimensional tiling in the Lee metric, European J. Combin. 19 (1998) 567–572] that there is no tiling of the three-dimensional space $\mathbb{R}^3$ with Lee spheres of radius at least 2. In particular, this verifies the Golomb–Welch conjecture for $n = 3$. Špacapan, [S. Špacapan, Non-existence of face-to-face four-dimensional tiling in the Lee metric, European J. Combin. 28 (2007) 127–133], using a computer-based proof, showed that the statement is true for $\mathbb{R}^4$ as well. In this paper we introduce a new method that will allow us not only to provide a short proof for the four-dimensional case but also to extend the result to $\mathbb{R}^5$. In addition, we provide a new proof for the three-dimensional case, just to show the power of our method, although the original one is more elegant. The main ingredient of our proof is the non-existence of the perfect Lee 2-error correcting code over $\mathbb{Z}$ of block size $n = 3, 4, 5$.

1. Introduction

Let $(\mathcal{C}, d)$ be a metric space. Then a code is any subset M of $\mathcal{C}$, $|M| \geq 2$. The elements of $\mathcal{C}$ will be called *words*, while elements of M will be referred to as *codewords*. The most common metric in coding theory is the Hamming metric. In this paper we deal with another frequently used metric, the so-called Lee metric (=the zig–zag metric, the Manhattan metric). The Lee metric d_L in $\mathbb{R}^n$ is given by $d_L(U, V) = \sum_{i=1}^{n} |u_i - v_i|$, where $U = (u_1, u_2, \ldots, u_n)$, $V = (v_1, v_2, \ldots, v_n)$.

As usual $\mathbb{Z}$ will stand for the set of integers. The perfect Lee t-error correcting code over $\mathbb{Z}$ of block size n, denoted $PL(n, t)$, is a set $M \subset \mathbb{Z}^n$ of codewords so that each word $A \in \mathbb{Z}^n$ is at Lee distance at most t from exactly one codeword in M. Since $PL(n, t)$ code can be seen as a partition of $\mathbb{Z}^n$ into spheres

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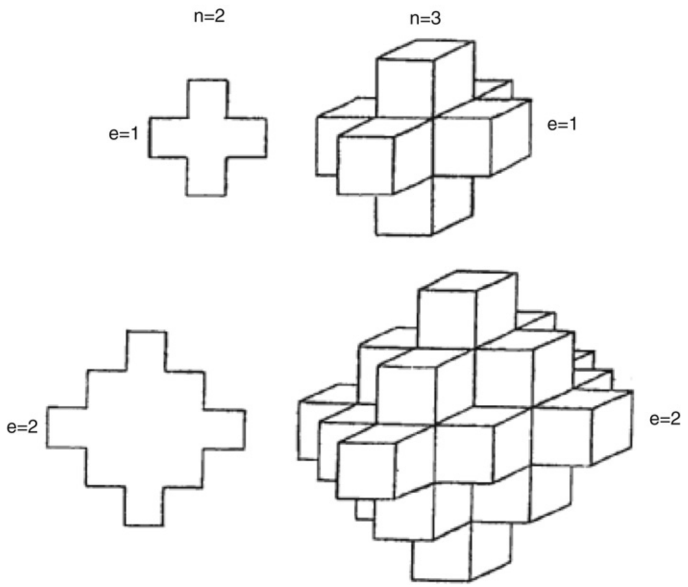

Fig. 1.

with radius t centered at codewords, only a small step is needed to get a geometrical interpretation of $PL(n,t)$ codes. Consider the space $\mathbb{R}^n$. The *n-cube* centered at $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the set: $C(X) = \{Y = (y_1, \ldots, y_n), y_i = x_i + \alpha_i$, where $-\frac{1}{2} \leq \alpha_i \leq \frac{1}{2}\}$. By a Lee sphere of radius r in $\mathbb{R}^n$, $L(n,r)$, centered at O we understand the union of n-cubes centered at Y, where $d_L(O,Y) \leq r$, and $Y - O$ has integer coordinates. Finally, a Lee sphere of radius r in $\mathbb{R}^n$ centered at $X \in \mathbb{R}^n$ is a translation of $L(n,r)$ centered at O along the coordinate axes so that O is mapped on X. Clearly, a $PL(n,t)$ code exists if and only if there is a tiling of $\mathbb{R}^n$ by Lee spheres of radius t. The Lee spheres $L(2,1)$, $L(2,2)$, $L(3,1)$, and $L(3,2)$ are depicted in Fig. 1.

The most famous and intensively studied problem in the area of Lee codes is the Golomb–Welch conjecture. In [3] it is shown that $PL(n,1)$ code exists for all $n \geq 1$, and $PL(2,t)$ code exists for all $t \geq 1$. In addition, it is proved there that there is no $P(3,2)$ code, and that there are no $PL(3,a_n)$ codes, where $a_n \to \infty$ is not explicitly specified. The authors conjectured:

Conjecture 1. *Golomb–Welch: There are no $PL(n,t)$ codes for $n > 2$ and $t > 1$.*

There are many results supporting the conjecture. The strongest one was proved by Post [8]:

Theorem 2. *$PL(n,t)$ codes do not exist for $n = 3$ and $t \geq 2$; for $4 \leq n \leq 5$ and $t \geq n-2$; and for $n \geq 6$ and $t \geq \frac{\sqrt{2}}{2}n - \frac{1}{4}(3\sqrt{2}-2)$.*

In the final remark Post states that, by using a computer to evaluate coefficients of the Taylor series of a suitable function, it is possible to show that there are no perfect t-error correcting codes for $6 \leq n \leq 130$ and $t \geq \frac{1}{16}(9n-15)$; and for $131 \leq n \leq 305$ and $t \geq \frac{1}{16}(9n-14)$. The reader interested in the non-existence results for Lee codes over finite sets is referred to [1,2,7], and also to [9] for the size of the largest Lee codes over a finite set. It is speculated in [6] that the most difficult cases to prove in the Golomb–Welch conjecture are those for $t = 2$ because they are the threshold cases ($PL(n,1)$ codes do exist). The Golomb–Welch conjecture has been verified there for the two smallest opened cases:

Theorem 3. *There is no $PL(n,2)$ code for $n = 5$ and 6.*

Thus, the Golomb–Welch conjecture has been verified for all pairs (n, t) where $n \leq 6$.

In [4,5], the authors prove, for the three-dimensional case, a result even stronger than conjectured by Golomb and Welch. They formulate it in terms of tilings of $\mathbb{R}^3$ by Lee spheres. It is shown in [4] that:

Theorem 4. *There is no tiling of $\mathbb{R}^3$ with Lee spheres of radii at least two, even with different radii.*

Thus, as a special case, they showed that there is no $PL(3, t)$ code for any $t \geq 2$. The authors provide a very elegant "a picture says it all" proof. Yet, a stronger result is proved in [5], a sequel to [4], where it is shown that there is no tiling of $\mathbb{R}^3$ with Lee spheres if radius of at least one sphere is greater than one. Recently, Špacapan [10] extended Theorem 4 to the four-dimensional case.

Theorem 5. *There is no tiling of $\mathbb{R}^4$ with Lee spheres of radii at least two, even with different radii.*

The both proofs in [4] and in [10] have one feature in common; they are "from scratch", they do not use any known result. On the other hand, the proof in [10] differs essentially from that one in [4]. It requires checking a large amount of cases and therefore it is computer-based.

In this paper we introduce a new method which provides a relatively short proof, not aided by a computer, for Theorem 5, but also for the five-dimensional case. We will give a new, short proof for Theorem 4 as well, although the original one given in [4] is more elegant, just to show the power of our method. The proof does not split into cases for $n = 3$, and considers only two case for $n = 4$, and three cases for $n = 5$. Unlike the proofs in [4] and in [10], our method is based on a known result, namely on the non-existence of the perfect Lee 2 -error correcting codes over $\mathbb{Z}$ of block size $n = 3, 4$, and 5. Thus, "as a by-product", our method provides some evidence that the most difficult cases in the Golomb–Welch conjecture are those for $t = 2$, because they imply, as a special case, the non-existence of $PL(n, t)$ for $3 \leq n \leq 5$, and $t \geq 3$. Our proof is "algebraic" in nature. Therefore we will first generalize the notion of the perfect Lee t-error correcting code. As usual, by a sphere $S = (W, r_W)$, centered at W and of radius r_W, we understand the set of all words $V \in \mathbb{Z}^n$ so that $d_L(W, V) \leq r_W$. For $V \in S$, we will also say that S covers V. The perfect Lee code over $\mathbb{Z}$ of block size n, denoted $PL(n)$, is a set $\mathcal{P}$ of spheres (W, r_W), $W \in \mathbb{Z}^n$, $r_W \geq 2$, so that each word in $\mathbb{Z}^n$ is covered by exactly one sphere in $\mathcal{P}$. The main theorem of the paper reads as follows:

Theorem 6. *There is no $PL(n)$ code for $3 \leq n \leq 5$.*

We believe that a further refinement of the method should provide a proof of the non-existence of $PL(n)$ at least for $n = 6$.

At the end of this introduction we mention a result which is related to the topic of this paper. A tiling of $\mathbb{R}^n$ by Lee spheres is called regular if neighboring spheres meet along entire $(n - 1)$-dimensional faces of the original cubes. It is shown in [4] and [5] that the results stated there hold even in the case if we admit non-regular tilings. At the first glance it seems obvious that there are no non-regular tilings of $\mathbb{R}^n$ by Lee spheres. However, in [11] Szabo proved the following surprising result:

Theorem 7. *There is a non-regular tiling of $\mathbb{R}^n$ if and only if $2n + 1$ is not a prime.*

2. $PL(n)$ codes for $3 \leq n \leq 5$.

In this section we prove the main result of the paper.

Throughout the proof words in $\mathbb{Z}^n$ will be denoted by upper case block letters, and their coordinates by the same lower case letter endowed with an index, e.g., a word W will have coordinates $(w_1, \ldots, w_n)$. Further, we drop subscript L when dealing with Lee metric, so the Lee distance will be denoted simply by d. The statement will be proved by contradiction. Suppose that there is a $PL(n)$ code $\mathcal{P}$, where $3 \leq n \leq 5$. By Theorems 2 and 3, there is no perfect Lee 2-error correcting code $PL(n, 2)$ for $3 \leq n \leq 5$ (note that Theorems 4 and 5 imply the statement for $n = 3$ and $n = 4$, respectively, as well). Thus, there is a sphere $S_0 = (A, r_A) \in \mathcal{P}$ so that $r_A \geq 3$. By a suitable translation of $\mathcal{P}$ we may assume that $A = (-r_A + 2, 0, \dots, 0)$. Consider the set $\mathcal{V}$ of words V with $d(V, A) = r_A + 1$, and $v_1 \geq 0$. Clearly, $V \in \mathcal{V}$ iff

$$\sum_{i=1}^{n} |v_i| = 3. \tag{1}$$

Indeed, $d(V, A) = \sum_{i=1}^{n} |v_i - a_i| = |v_1 - (2 - r_A)| + \sum_{i=2}^{n} |v_i| = v_1 + r_A - 2 + \sum_{i=2}^{n} |v_i| = r_A + 1$, and (1) follows. Therefore, each word V in $\mathcal{V}$ is either of type $[\pm 3]$, or of type $[\pm 2, \pm 1]$, or $[\pm 1^3]$.

To prove the non-existence of $PL(n)$ code for $3 \leq n \leq 5$, we show that it is impossible to cover all words in $\mathcal{V}$, that is, we show that there is no set of pairwise disjoint spheres (and disjoint from S_0), each of radius at least 2, covering all words in $\mathcal{V}$. To this extent, let $\mathcal{S} \subset \mathcal{P}$ be the set of all spheres in $\mathcal{P}$ which cover at least one word in $\mathcal{V}$. The words W so that $(W, r_W) \in \mathcal{P}$ will be called codewords, the words W so that $(W, r_W) \in \mathcal{S}$ will be called codewords in $\mathcal{S}$. Moreover, if a word V belongs to a sphere $S = (W, r_W) \in \mathcal{P}$, we will abuse slightly the language and sometimes instead of saying S covers V we will say that the codeword W covers V.

Now we state a series of statements which are rather simple but will be applied over and over in this proof, although not always explicitly referred to. By definition of $PL(n)$ code we get

Claim 8. *If W, Z are codewords then the spheres (W, r_W) and (Z, r_Z) are disjoint, that is, $d(W, Z) \geq r_W + r_Z + 1$.*

For any two words U, V, their Lee distance $d(V, W)$ is invariant with respect to adding the same integer to a coordinate, multiplying a coordinate by -1, or swapping the order of coordinates. Therefore:

Claim 9. *If $\mathcal{P}$ is a $PL(n)$ code, then* (i) *translating all codewords of $\mathcal{P}$,* (ii) *multiplying a coordinate of each codeword of $\mathcal{P}$ by -1,* (iii) *swapping the order of coordinates in all codewords of $\mathcal{P}$, results in a new $PL(n)$ code.*

The following claim plays a crucial role in the description of words in $\mathcal{V}$ covered by a codeword in $\mathcal{S}$.

Claim 10. *Let W, Z be codewords (not necessarily in $\mathcal{S}$), V be a word covered by W, and $d(Z, V) \leq r_Z + 2$. Then, for each coordinate i, it is either $z_i \leq v_i \leq w_i$, or $z_i \geq v_i \geq w_i$.*

Proof. By Claim 9, we may assume that $Z = O = (0, 0, \dots, 0)$. Then in fact we need to prove that $v_i w_i \geq 0$, and $|w_i| \geq |v_i|$ for all i. Since W covers V we have $r_W \geq d(W, V) = \sum_{i=1}^{n} |w_i - v_i|$. The spheres (W, r_W) and (O, r_O) are disjoint, therefore $d(W, O) = \sum_{i=1}^{n} |w_i| \geq r_W + r_O + 1 \geq \sum_{i=1}^{n} |w_i - v_i| + (d(O, V) - 2) + 1$ (by assumption $d(O, V) \leq r_O + 2$). After simple rearrangements we get $\sum_{i=1}^{n} |w_i| - \sum_{i=1}^{n} |v_i| \geq \sum_{i=1}^{n} |w_i - v_i| - 1$. Trivially $|a| - |b| \leq |a - b|$ for all a, b, with equality iff $ab \geq 0$, and $|a| \geq |b|$. To complete the proof it suffices to note that for $ab < 0$ it is $|a| - |b| < |a - b| - 1$. Thus $v_i w_i \geq 0$, and $|w_i| \geq |v_i|$ for all $i \leq n$. The claim follows. ∎

As an immediate corollary of the above claim we get:

Claim 11. *If W is a codeword in $\mathcal{S}$ then $\sum_{i=1}^{n} |w_i| \geq 5$, and $w_1 \geq 0$.*

Proof. By definition of $\mathcal{S}$ there is a word $V \in \mathcal{V}$ covered by W. For each word V in $\mathcal{V}$ we have $d(A, V) = r_A + 1$. Since $v_1 \geq 0$ for all words in $\mathcal{V}$, by Claim 10, $w_1 \geq 0$ as well. In addition, $d(W, A) = (w_1 - (2 - r_A)) + \sum_{i=2}^{n} |w_i| \geq r_A + r_W + 1 \geq r_A + 2 + 1$, that is, $\sum_{i=1}^{n} |w_i| \geq 5$. The claim follows. ∎

The following corollary is the most frequently used statement of all claims given here. It provides a simple but very useful description of all words V in $\mathcal{V}$ covered by a codeword W in $\mathcal{S}$. We point out that the description involves only coordinates of W but not the radius r_W of the sphere $S = (W, r_W) \in \mathcal{S}$.

Claim 12. *Let W be a codeword in $\mathcal{S}$. Then W covers a word V in $\mathcal{V}$ if and only if $v_i w_i \geq 0$, $|w_i| \geq |v_i|$ for all $i = 1, \dots, n$.*

Proof. The necessary part follows from Claim 10. Indeed, for each word V in $\mathcal{V}$ we have $d(A, V) = r_A + 1$, therefore, for $i \geq 2$, either $0 \leq v_i \leq w_i$, or $0 \geq v_i \geq w_i$, i.e., $v_i w_i > 0$, and $|w_i| \geq |v_i|$. By definition, $v_1 \geq 0$, and by Claim 10, $w_1 \geq v_1$. As to the sufficiency part, it suffices to note that W being a codeword in $\mathcal{S}$ implies that W covers at least one word in $\mathcal{V}$, and that if $V, V' \in \mathcal{V}$ are two words fulfilling the condition in the claim then $d(W, V) = d(W, V')$. ■

Example. Let $W = (1, 7, -1, -2)$ be a codeword in $\mathcal{S}$. Then Claim 12 implies that W covers in $\mathcal{V}$ only one word of type $[\pm 3]$, namely $(0, 3, 0, 0)$, the following words of type $[\pm 2, \pm 1]$: $(1, 2, 0, 0)$, $(0, 2, -1, 0)$, $(0, 2, 0, -1)$, $(1, 0, 0, -2)$, $(0, 1, 0, -2)$, and $(0, 0, -1, -2)$, and four words of type $[\pm 1^3]$: $(1, 1, -1, 0)$, $(1, 1, 0, -1)$, $(1, 0, -1, -1)$, and $(0, 1, -1, -1)$.

The following three relations are essential for the proof of our theorem. Let $C(t)$ be the set of codewords in $\mathcal{S}$ having t non-zero coordinates. Further, let $C_i(t)$ be a subset of $C(t)$ that contains codewords W so that $|\{j, \mid w_j| > 1\}| = i$. Set $|C(t)| = c(t)$ and $|C_i(t)| = c_i(t)$.

There are $2n - 1$ words of type $[\pm 3]$ in $\mathcal{V}$ (note that for V in $\mathcal{V}$ it is $v_1 \geq 0$). By Claim 11, for each codeword W in $C_1(3)$ there is i, $1 \leq i \leq n$, so that $|w_i| \geq 3$, and consequently, by Claim 12, W covers one of those words of type $[\pm 3]$; thus

$$c_1(3) \leq 2n - 1. \tag{2}$$

Further, there are $8\binom{n}{2} - 4(n-1)$ words of type $[\pm 2, \pm 1]$ in $\mathcal{V}$. Hence

$$4c(3) - 2c_1(3) + 3c(4) \leq 8\binom{n}{2} - 4(n-1) \tag{3}$$

as, by Claim 12, each codeword in $C_1(3)$ covers exactly two of those words, each codeword in $C(3) - C_1(3)$ at least 4 of them, and each codeword in $C(4)$ at least 3 of them (for each codeword W in $C(4)$ there is at least one i so that $|w_i| \geq 2$, see Claim 11). Finally, there are $8\binom{n}{3} - 4\binom{n-1}{2}$ words of type $[\pm 1^3]$ in $\mathcal{V}$, which leads to

$$\sum_{i=3}^{n} \binom{i}{3} C(i) = 8\binom{n}{3} - 4\binom{n-1}{2} \tag{4}$$

because, by Claim 12, each codeword in $C(i)$ covers exactly $\binom{i}{3}$ words of type $[\pm 1^3]$.

Clearly, there are many solutions of Eqs. (2)–(4) in the non-negative integers. A solution, which corresponds to a perfect Lee code over $\mathbb{Z}$, will be called a feasible solution. So our aim will be to show that there is no feasible solution of these equations.

2.1. $n = 3, 4$

Here we provide alternative proofs to Theorems 4 and 5. We start with a statement, which plays an essential role in the closing argument of those proofs.

Theorem 13. *For $n = 3$, there is no codeword $W \in C_2(2)$ so that $w_1 = 0$. For $n = 4$, there is no codeword $W \in C_2(2)$, so that $w_1 = 2$.*

Proof. The statement will be proved by contradiction. We start with the case $n = 4$. Suppose that $W \in C_2(2)$, where $w_1 = 2$. Then there is i, $2 \leq i \leq 4$, so that $|w_i| \geq 3$, (see Claim 11). By Claim 9 we assume, without loss of generality, that $w_2 \geq 3$, and $w_i = 0$, for $3 \leq i \leq 4$. Let, for $j = 1, 2, 3$, $\mathcal{M}_j$ be a set of words given by $\mathcal{M}_1\{V, v_1 = 1, v_2 = 2, \sum_{i=3}^{4} |v_i| = 1\}$, $\mathcal{M}_2 = \{V, v_1 = 1, v_2 = 1, \sum_{i=3}^{4} |v_i| = 1\}$, $\mathcal{M}_3 = \{V, v_1 = 0, v_2 = 2, \sum_{i=3}^{4} |v_i| = 1\}$. Clearly, for all j, $\left|\mathcal{M}_j\right| = 4$. The following claim is crucial for the proof of this theorem.

Claim 14. *Let Z be a codeword covering a word $V \in \mathcal{M}_j$, $1 \leq j \leq 3$. Then $z_1 = v_1$, and $z_2 = v_2$.*

Proof of Claim. We recall that $A = (2 - r_A, 0, 0, 0)$, $r_A \geq 3$, is a codeword. Let $V \in \mathcal{M}_j$, $1 \leq j \leq 3$. Then $d(A, V) \leq r_A + 2$, and therefore, by Claim 10, $z_1 \geq v_1 \geq 0 > 2 - r_A$, and $z_2 \geq v_2 \geq 0$. On the other hand, W is a codeword in $\mathcal{S}$, so, by Claim 12, W covers both words $(2, 1, 0, 0)$ and $(1, 2, 0, 0)$. Hence $r_W = d(W, (2, 1, 0, 0)) = w_2 - 1$ (note that r_W cannot be $> w_2 - 1$ as then the spheres (A, r_A) and (W, r_W) would intersect). Therefore, $d(W, V) = \sum_{i=1}^4 |w_i - v_i| = (w_1 - v_1) - (w_2 - v_2) + 1 \leq 2 + w_2 - 2 + 1 = w_2 + 1 = r_w + 2$ and, again by Claim 10, $z_1 \leq v_1 \leq w_1$, and $z_2 \leq v_2 \leq w_2$. The proof follows. ∎

Now we will classify codewords Z covering words in $\bigcup_{i=1}^3 \mathcal{M}_i$. Set, for $j = 1, 2$, $A_j = \{Z, Z$ is a codeword covering a word in $\bigcup_{j=1}^3 \mathcal{M}_j$, and $|\{i, i \geq 3, z_i \neq 0\}| = j$. Put $a_i = |A_i|$. We prove two inequalities on a_i.

It is clear that each codeword from A_i covers i words in $\bigcup_{j=1}^3 \mathcal{M}_j$. Indeed, since $\bigcup_{j=2}^3 \mathcal{M}_j \subset \mathcal{V}$, the statement in this case follows from Claim 12. For a codeword covering a word in $\mathcal{M}_1$, the assumption that $U = (1, 2, d, 0)$, $d \neq 0$, covers a word $(1, 2, 0, c)$, $|c| = 1$, implies that spheres (U, r_U) and (W, r_W) intersect. The other part is obvious. As $\left|\bigcup_{j=1}^3 \mathcal{M}_j\right| = 12$, we get

$$2a_2 + a_1 = 12. \tag{5}$$

To finish the proof of the theorem we prove that

$$a_1 + a_2 \leq 4 \tag{6}$$

which contradicts (5).

First we point out that if Z covers a word $V \in \bigcup_{j=1}^3 \mathcal{M}_j$ then

$$|z_3| + |z_4| \geq 3. \tag{7}$$

Indeed, if V is in $\mathcal{M}_2 \cup \mathcal{M}_3$, then the inequality follows from Claim 11; for a word V in $\mathcal{M}_1$ it can be routinely checked that $|z_3| + |z_4| < 3$ implies that the spheres (Z, r_Z) and (W, r_W) intersect because $r_W \geq w_2 - 1$ and $d(Z, W) \leq w_2 + 1$ in this case. Let Z, Z' be a codeword in $A_1 \cup A_2$ covering a word $V, V' \in \bigcup_{i=1}^3 \mathcal{M}_i$, respectively. To prove (6) it is sufficient to show, with respect to (7), that if Z, $Z' \in A_1 \cup A_2$, and $z_j z_j' > 0, j \geq 3$, then either $\left|z_j\right| = 1$, or $\left|z_j'\right| = 1$. Assume wlog $j = 3$. Suppose by contradiction that $|z_3| > 1$, and $\left|z_3'\right| > 1$. We have $z_3 z_3' > 0$, and $\min\{|z_3|, \left|z_3'\right|\} \geq 2$, which yields

$$\left|z_3 - z_3'\right| \leq |z_3| + \left|z_3'\right| - 4. \tag{8}$$

Further, the spheres (Z, r_Z) and $(Z', r_{Z'})$ are disjoint, thus we get $d(Z, Z') = \sum_{i=1}^4 \left|z_i - z_i'\right| \geq r_Z + r_{Z'} + 1 \geq d(Z, V) + d(Z', V') + 1 = \sum_{i=1}^4 |z_i - v_i| + \sum_{i=1}^4 \left|z_i' - v_i'\right| + 1$. Applying Claim 14 into $\sum_{i=1}^4 \left|z_i - z_i'\right| \geq \sum_{i=1}^4 |z_i - v_i| + \sum_{i=1}^4 \left|z_i' - v_i'\right| + 1$ provides $\left|v_1 - v_1'\right| + \left|v_2 - v_2'\right| + \sum_{i=3}^4 \left|z_i - z_i'\right| \geq \sum_{i=3}^4 |z_i - v_i| + \sum_{i=3}^4 \left|z_i' - v_i'\right| + 1$; now using (8) and obvious $\left|z_4 - z_4'\right| \leq |z_4| + \left|z_4'\right|$ yields $\left|v_1 - v_1'\right| + \left|v_2 - v_2'\right| \geq 3$. However, $\left|v_1 - v_1'\right| \leq 1$, and $\left|v_2 - v_2'\right| \leq 1$, a contradiction, and (6) follows.

Finally, let $n = 3$. Assume wlog that $W = (0, a, b) \in C(2)$, where $a \geq 2, b \geq 3$. Consider the words $V_1 = (1, 1, 1)$, and $V_2 = (1, 1, 2)$. By the same argument as in the proof of Claim 14, we get that if a codeword Z_i covers $V_i, i = 1, 2$, then $Z_1 = (c, 1, 1), Z_2 = (d, 1, 2)$, where, by Claim 11, $c \geq 3$, and $d \geq 2$ (otherwise the spheres (A, r_A) and (Z_2, r_{Z_2}) would intersect). However, then the spheres (Z_1, r_{Z_1}) and (Z_2, r_{Z_2}) intersect. Indeed, $r_{Z_1} \geq c - 1$ while $r_{Z_2} \geq d - 1$, but $d(Z_1, Z_2) = |c - d| + 1 < (c - 1) + (d - 1) + 1 = r_{Z_1} + r_{Z_2} + 1$. The proof of the theorem is complete. ∎

Theorem 15. *There is no PL(3) code.*

Proof. Putting $n = 3$ into Eq. (4), and taking into account that in this case $C(4) = C(5) = 0$ we get $c(3) = 4$.

First of all we show that there is a codeword $W \in C_1(3)$ so that $w_1 = 1$. Indeed, otherwise $c(3) - c_1(3) \geq 3$, and the codewords in $C(3) - C_1(3)$ would have to cover twice some word in $\mathcal{A} = \{V,$

V is of type $[\pm 2, \pm 1]$, $v_1 = 1\}$. So, suppose wlog that $W = (1, d, 1)$, where $d \geq 3$. Let $Z = (a, b, c)$ be a codeword covering the word $(0, 2, -1)$. Then, by Claim 12, $b \geq 2$ and $c \leq -1$; in addition, $a = 0$, otherwise the word $(1, 2, 0)$ would be covered by both W and Z. If $b > 2$ then Z would cover $(0, 3, 0)$, which leads to a contradiction because $(0, 3, 0)$ would be covered by both W and V. Hence, $b = 2$, and by Claim 11, $|c| \geq 3$, that is, $Z \in C_2(3)$ with $z_1 = 0$, contradicting Theorem 13. ■

Theorem 16. *There is no PL(4) code.*

Proof. For the reader's convenience we state that, for $n = 4$, the equations (2)–(4) turn into

$$c_1(3) \leq 7$$
$$4c(3) - 2c_1(3) + 3c(4) \leq 36$$
$$c(3) + 4c(4) = 20.$$

First of all we show that if there were a feasible solution of Eqs. (2)–(4) then $3 \leq c(4) \leq 4$. Let $\mathcal{A} = \{V, V$ is a word of type $[\pm 1^3]$, $v_1 = 0\}$. Clearly, $|\mathcal{A}| = 8$. By Claim 12, each word in $C(3) \cup C(4)$ covers at most one word in $\mathcal{A}$, thus $c(3) + c(4) \geq 8$. Therefore, from (4), $c(4) \leq 4$. On the other hand, it is easy to see that, for $c(4) \leq 2$, $m = \min(4c(3) - 2c_1(3) + 3c(4))$ is attained when $c_1(3)$ and $c(4)$ are maximum possible, i.e. $c_1(3) = 7$, and $c(4) = 2$. Thus, $m = 48 - 14 + 6 = 40 > 36$, contradicting (3). We consider two cases.

I. $c(4) = 4$. By Claim 9 we may assume that there is a codeword $Z \in C(4)$ with $z_i > 0$ for all $i = 1, \ldots, 4$. Then the coordinates of the four codewords in $C(4)$ have the following signs: $(+, +, +, +)$, $(+, +, -, -)$, $(+, -, +, -)$, and $(+, -, -, +)$. Let $\mathcal{B} = \{V, V$ is of type $[\pm 2, \pm 1]$, $v_1 = 2\}$. Clearly, $|\mathcal{B}| = 6$. At most one codeword W in $C(4)$ has the property that $w_1 > 1$, otherwise, by Claim 12, a word V of type $[\pm 2, \pm 1]$ with $v_1 = 2$ and $v_j = 1$, for some j, $2 \leq j \leq 4$, would be covered by two codewords from $C(4)$. Thus, codewords in $C(4)$ cover at most three words in $\mathcal{B}$. There are twelve words V of type $[\pm 1^3]$ with $v_1 \neq 0$. Each codeword in $C(4)$ covers three of them. As $c(4) = 4$, all of those twelve words are covered by codewords in $C(4)$. Hence, by Claim 12, for each codeword $W \in C(3)$ we have $w_1 = 0$. This in turn implies that no word in $\mathcal{B}$ is covered by a codeword in $C(3)$. Hence, at least three words in $\mathcal{B}$ have to be covered by codewords Z in $C(2)$, with $z_1 > 1$. By Claim 12, each codeword in $C(2)$ covers at most one word from $\mathcal{B}$, so there are at least three codewords in $C(2)$ covering a word in $\mathcal{B}$. Since at most one of them has its first coordinate ≥ 3 (otherwise the word $(3, 0, 0, 0)$ would be covered by more than one codeword), there is a codeword U in $C(2)$ with $u_1 = 2$. However, this contradicts Theorem 13.

II. $c(3) = 3$. Assume wlog that the coordinates of codewords in $C(4)$ have the following signs: $(+, +, +, +)$, $(+, +, -, -)$, and $(+, -, +, -)$. These three codewords in $C(4)$ cover nine out of twelve codewords V of type $[\pm 1^3]$ with $v_1 = 1$. To cover the three remaining words V, there have to be, by Claim 12, codewords U_i, $i = 1, 2, 3$, in $C(3)$ with coordinates $(+, 0, -, +)$, $(+, -, 0, +)$, and $(+, -, -, 0)$, respectively. Moreover, there has to be a codeword U_4 in $C(3)$ with coordinates $(0, -, -, +)$ to cover the word $(0, -1, -1, 1)$. It is not difficult to check that, to avoid some word of type $[\pm 2, \pm 1]$ to be covered by two codewords U_i, that $U_i \in C_1(3)$ for $i = 1, \ldots, 4$. Thus, there is i, $1 \leq i \leq 4$, so that the first coordinate of U_i is ≥ 3. Now we focus on the set of words $\mathcal{B} = \{V, V$ is of type $[\pm 2, \pm 1]$, $v_1 = 2\}$. As in the case $c(4) = 4$, there is in $C(4)$ at most one codeword W with $w_1 \geq 2$. In addition, there is exactly one codeword U in $C(3)$ with $u_1 \geq 2$ (in fact for this codeword its first coordinate ≥ 3, as mentioned above). Therefore, at most 5 words from $\mathcal{B}$ are covered by codewords in $C(3) \cup C(4)$; hence there has to be a codeword $Z \in C(2)$, covering a word in $\mathcal{B}$. To avoid covering the word $(3, 0, 0, 0)$ twice, it has to be $z_1 = 2$, contradicting Theorem 13. ■

2.2. $n = 5$

In order to facilitate our discussion we introduce more notions and notation. Two words U and V are said to be sign equivalent in the ith coordinate if $u_i v_i > 0$. We will deal very often with a set of words that are sign equivalent in some coordinate. For each coordinate we have two such sets. To simplify the language we will introduce the notion of the *signed coordinate*. For the rest of the paper by the set of signed coordinates we will understand the set $I = \{+1, \ldots, +5, -2, \ldots, -5\}$ (we recall that by definition it is $v_1 \geq 0$ for each word V in $\mathcal{V}$; and by Claim 12 it is $w_1 \geq 0$ for each codeword W in $\mathcal{S}$). Let A be a set of words, and i, j, $|i| \neq |j| \in I$. Then $A_i(A_{ij})$ is the set of all words in A so that $i.w_{|i|} > 0$ ($iw_{|i|} > 0, jw_{|j|} > 0$). In other words, A_i is the set of words in $\mathbb{Z}^n$ that are pairwise sign equivalent in the $|i|$th coordinate, and their common sign in the $|i|$th coordinate coincide with the sign of i. We note that no two codewords in $\mathcal{S}$ are sign equivalent in three coordinates because they would cover the same word of type $[\pm 1^3]$, see Claim 12.

The Eqs. (2)–(4) describe "global" properties of parameters $c(t)$. The following three theorems describe their "local" properties.

Theorem 17. *It is* $\sum_{t=3}^{5} \binom{t-1}{2} |C(t)_{+1}| = 24$, *and for each* $i \in I, i \neq +1$, *we have* $\sum_{t=3}^{5} \binom{t-1}{2} |C(t)_i| = 18$. *Consequently,* $|C(3)_i| \equiv 0 \pmod 3$ *for all* $i \in I$.

Proof. Let T be the set of all words in $\mathcal{V}$ of type $[\pm 1^3]$. Clearly, $|T_{+1}| = 4\binom{4}{2} = 24$, and for $i \in I$, $i \neq +1$, it is $|T_i| = 4\binom{4}{2} - 6 = 18$. Further, each word in T_i is covered by exactly one word from $\bigcup_{t=3}^{5} C(t)_i$, and, by Claim 12, each codeword from $C(t)_i$ covers $\binom{t-1}{2}$ of them. We are done with the first part of the statement. The second part follows trivially from the fact that $3|\binom{t-1}{2}$ for $t = 4, 5$. ■

Theorem 18. *For each* $i, j \in I$, $+1 \in \{i, j\}$, $|i| \neq |j|$, $\sum_{t=3}^{5}(t-2)C(t)_{ij} = 6$; *and for each* $i, j \in I$, $+1 \notin \{i, j\}$, $\sum_{t=3}^{5}(t-2)C(t)_{ij} = 5$. *Consequently,* $\left|C(3)_{ij}\right|$ *and* $\left|C(5)_{i,j}\right|$ *have the same parity for* $+1 \in \{i, j\}$, *and the opposite parity for* $+1 \notin \{i, j\}$.

Proof. As in the previous proof, let T be the set of all words of type $[\pm 1^3]$. Then, for $|i| \neq |j|$, $\left|T_{ij}\right| = 2(5-2)$ for $+1 \in \{i, j\}$, otherwise $\left|T_{ij}\right| = 5$. Each word in T_{ij} is covered by exactly one codeword from $\bigcup_{t=3}^{5} C(t)_i$. As, by Claim 12, each codeword from $C(t)_{ij}$ covers $t-2$ words in T_{ij}, the main part of the statement follows. The second part is obvious. ■

We state one more theorem that will significantly decrease the number of feasible solutions of (2)–(4).

Theorem 19. *It is* $|C(3)_{+1}| \leq 9$. *For* $i \in I, i \neq +1$, $|C(3)_i| \leq 3$ *when* $|C(5)_i| = 0$, *and* $|C(3)_i| \leq 6$ *when* $|C(5)_i| = 1$.

Proof. Let $i \in I$. Further, set $\mathcal{A}_i = \{V, v_{|i|} = \pm 1\}$, and $\mathcal{B}_i = \{V, v_{|i|} = \pm 2\}$. Clearly, $|\mathcal{A}_i| = |\mathcal{B}_i| = 7$ for each $i \in I, i \neq +1$, and $|\mathcal{A}_{+1}| = |\mathcal{B}_{+1}| = 8$. Let a be the number of words in $\mathcal{A}_i$ covered by codewords in $C(3)_i$. It is easy to see that if $W \in C(3)_i$ does not cover any word in $\mathcal{A}_i$ then $W \in C_1(3)_i$ and $\left|w_{|i|}\right| \geq 3$. However, there is at most one such codeword in $C(3)_i$; thus $a \geq |C(3)_i| - 1$. This in turn implies, $|C(3)_{+1}| \leq 9$ as $|\mathcal{A}_{+1}| = 8$.

Let $i \in I, i \neq +1$. Put $D = \{W, W \in \bigcup_{t=3}^{5} C(t)_i, W$ covers a word in $\mathcal{B}_i\}$. Now we state some bounds on D. Let $U \in C(t)_i$, $3 \leq t \leq 5$, $|u_{|i|}| > 1$ (i.e., $U \in C$). Then U covers $t-1$ words in $\mathcal{B}_i$. Thus, $|\mathcal{B}_i| = 7$ implies $|D| \leq 3$, and consequently $|D| = 3$ implies $\left|\bigcup_{t=4}^{5} C(t)_i \cap D\right| \leq 1$. Further, if $\left|\bigcup_{t=4}^{5} C(t)_i \cap D\right| = 2$ then $C(3)_i \cap D = \emptyset$. Clearly, if $W \in C(4)_i$ but not in B then W covers at least one word in $\mathcal{A}_i$ (by Claim 11 there is an index j so that $\left|w_j\right| \geq 2$). Therefore, $|\mathcal{A}_i| \geq a + |C(4)_i| - |C(4)_i \cap D| \geq |C(3)_i| + |C(4)_i| - 2$. Since $|\mathcal{A}_i| = 7$, we get $|C(3)_i| + |C(4)_i| \leq 9$. By Theorem 17, $|C(3)_i| + 3\,|C(4)_i| = 18 - 6\,|C(5)_i|$, and $|C(3)_i| \equiv 0 \pmod 3$. Hence $|C(3)_i| \leq 3$ for $|C(5)_i| = 0$, and $|C(3)_i| \leq 6$ for $|C(5)_i| = 1$. The proof is complete. ■

We are ready to prove the main result of the paper:

Theorem 20. *There is no PL*(5) *code.*

Proof. As in the previous cases, for the reader's convenience we state that, for $n = 5$, the Eqs. (2)–(4) have the form

$$c_1(3) \le 9$$
$$4c(3) - 2c_1(3) + 3c(4) \le 64$$
$$c(3) + 4c(4) + 10c(5) = 56.$$

We point out that $c(5) \le 2$. Indeed, by Claim 11, for each $W \in C(5)$ it is $w_1 \ge 0$, which in turn implies $|C(5)_{+1}| \le 2$, i.e. $c(5) \le 2$, otherwise some word of type $[\pm 1^3]$ would be covered by two codewords from $C(5)_{+1}$. We will consider three cases.

I. $c(5) = 2$. Let $W, Z \in C(5)$. By Claim 9 we assume wlog $w_i > 0$ for all i. We will consider two subcases with respect to the number of coordinates in which W and Z are sign equivalent.

Ia. Assume first that $z_i < 0$ for $2 \le i \le 5$. Then $|C(4)_{+1}| = 0$, otherwise there would be a word of type $[\pm 1^3]$ covered by two codewords. Thus, by Theorem 17, $|C(3)_{+1}| = 12$, which contradicts Claim 11.

Ib. There is i, $2 \le i \le 5$, so that $z_i > 0$. By Claim 9, we assume $z_2 > 0$, and $z_i < 0$ for $3 \le i \le 5$. It is easy to see that for each $U \in C(4)$ it is $u_2 < 0$. Set $A = \{W, W \in C(4), w_1 \ne 0\}$, and $B = \{W, W \in C(4), w_1 = 0\}$. By Theorem 19, $|C(3)_{-2}| \le 3$, which in turn implies $|C(4)_{-2}| \ge 5$. However, $c(4) = |C(4)_{-2}|$, thus $|A| + |B| \ge 5$. Further, for $|i| \ge 3$, $|j| \ge 3$, $\left|C(4)_{ij}\right| \le 1$, otherwise some word of type $[\pm 1^3]$ would be covered by two codewords. There are 6 pairs of indices i, j, with $|i| \ge 3$, $|j| \ge 3$, and $|i| \ne |j|$. Each codeword $U \in A$ belongs to $C(4)_{ij}$ for one of those six pairs of signed coordinates, and each codeword in B to two of those six pairs; thus $|A| + 2\,|B| \le 6$. Finally, for $|i| \ge 3$, it is $|A_i| \le 1$, otherwise some word of type $[\pm 1^3]$ would be covered by two codewords in A_i; hence $|A| \le 3$. However, this contradicts $|A| + |B| \ge 5$ because $|A| + 2\,|B| \le 6$.

II. $c(5) = 1$. Let $W \in C(5)$. As in case **I**, we assume $w_i > 0$ for all i. By Theorem 18, $\left|C(3)_{+1,i}\right|$ is odd for every $i \in \{+2, +3, +4, +5\}$, hence $|C(3)_{+1}| \ge 6$, as $|C(3)_i| \equiv 0 \pmod 3$, see Theorem 17. Further, again by Theorem 18, $\left|C(3)_{i,j}\right|$ is odd if $+1 \notin \{i, j\}$ and at least one of i and j is in $\{-2, \dots, -5\}$. Therefore, $|C(3)_i| \ge 3$ for all signed indices $i \in I$. Moreover, for $i \in \{-2, \dots, -5\}$, if $\left|C(3)_{+1,i,}\right| > 0$ then $|C(3)_i| \ge 6$ as then there are at least seven indices $j \in I$ for which $\left|C(3)_{i,j}\right| > 0$. No codeword in $C(3)_{+1}$ has all its coordinates non-negative. Hence, $\left|C_{(3)_{+1}}\right| \ge 6$ implies that there are at least three indices i in $\{-2, \dots, -5\}$ for which $|C(3)_i| \ge 6$, contradicting Theorem 19 because $c_i(5) = 0$.

III. $c(5) = 0$. By Theorem 18, $\left|C(3)_{ij}\right|$ is odd for each $i, j \in I$, $|i| \ne |j|$, $+1 \notin \{i, j\}$. Thus $|C(3)_i| \ge 3$, and by Theorem 19, $|C(3)_i| = 3$. By (4), we get $c(3) \equiv 0 \pmod 4$. Hence $|C(3)_{+1}| \equiv 0 \pmod 4$. As $|C(3)_{+1}| = 12$ contradicts Theorem 19, we get $|C(3)_{+1}| = 0$. Thus $c(3) = 8$, and by (4), $c(4) = 12$. Moreover, $|C(3)_{+1}| = 0$ implies $|C(4)_{+1}| = 8$. Therefore there are four codewords $U \in C(4)$ with $u_1 = 0$. It is easy to see that there have to be two codewords U_1, U_2 among those four codewords which are sign equivalent in two coordinates, say i, j. (In fact this would be true even if there were only three codewords $U \in C(4)$ with $u_1 = 0$}. As $|C(3)_{+1}| = 0$, there has to be a codeword $W \in C(4)_{+1}$ that is sign equivalent in the ith and jth coordinate with both U_1 and U_2. Let k, $2 \le k \le 5$, $k \notin \{i, j\}$ be a coordinate so that $w_k \ne 0$. Clearly, U_1 and U_2 are not sign equivalent in k, as they would be sign equivalent in three coordinates, thus one of them is sign equivalent with W in k as well, which again implies that there are two codewords sign equivalent in three coordinates, a contradiction. ■

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